# On integration of the closed KdV dressing chain

Yu. Brezhnev†, S. B. Leble‡,

‡Faculty of Applied Physics and Mathematics

Technical University of Gdan'sk,

ul. G.Narutowicza, 11/12 80-952, Gdan'sk-Wrzeszcz, Poland,

email leble@mifgate.pg.gda.pl

and

†‡Kaliningrad State University, Theoretical Physics Department,

Al.Nevsky st.,14, 236041 Kaliningrad, Russia.

**Abstract**

The dressing chain equations for the second order Sturm-Liouville differential operators is integrated. The simplest closure at the third step is linked to the spectral curve of the genus 1 and the explicit solution in elliptic Weierstrass functions is given. Dependence on integrals that enter the bihamiltonian structure is specified.

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## 1 Introduction

This introduction is a citation from the paper [1]. The Darboux transformation scheme of dressing you can found at [2, 3, 4], the chains were introduced at [5], studied at [6, 7]. Consider a dressing chain for a Sturm-Liouville differential operator

$$(\sigma_n + \sigma_{n+1})_x = \sigma_n^2 - \sigma_{n+1}^2 + \alpha_n \tag{1}$$

It is connected to the the linear Schrödinger equation

$$\Psi_{xx} = (q - \lambda)\Psi, \tag{2}$$

by $q_n = \sigma_{n_x} + \sigma_n^2 + \mu_n,\ \ \alpha_n = \mu_n - \mu_{n+1}$. Let us close the chain (1) by the condition

$$\sigma_i \equiv \sigma_{i+N},\ \mu_i \equiv \mu_{i+N}, \tag{3}$$

and assume that $\sum_{i=1}^{N} \alpha_i = 0$, then we have finite dimensional dynamical system

$$(\sigma_i + \sigma_{i+1})_x = \sigma_i^2 - \sigma_{i+1}^2 + \mu_i - \mu_{i+1},\ \ i = 1, ..., N. \tag{4}$$

As was shown by Veselov and Shabat [1], for $N = 2n+1$, it is bi-Hamiltonian system of the following form in $g_i = f_i + f_{i+1}$ coordinates:

$$\pi_\lambda dh_\lambda = 0 \Leftrightarrow \begin{array}{l} \pi_0 dh_0 = 0 \\ \pi_0 dh_1 = K_1 = \pi_1 dh_0 \\ \pi_0 dh_2 = K_2 = \pi_1 dh_1 \\ \qquad\qquad \vdots \\ \pi_0 dh_n = K_n = \pi_1 dh_{n-1} \\ \qquad\qquad 0 = \pi_1 dh_n, \end{array} \tag{5}$$

$$\{g_i, g_{i-1}\}_{\pi_0} = 1, \tag{6}$$

$$\begin{array}{rcl} \{g_i, g_j\}_{\pi_1} & = & (-1)^{j-i} g_i g_j, \qquad j \neq i \pm 1, \\ \{g_i, g_{i-1}\}_{\pi_1} & = & g_i g_{i-1} + \beta_i \end{array} \tag{7}$$

and the generating function for $h_i$ is

$$\begin{array}{rcl} \tau_N & = & \left[ \prod_{j=1}^{N} \left( 1 + \zeta_{j+1} \frac{\partial^2}{\partial g_j \partial g_{j+1}} \right) \right] \prod_{k=1}^{N} g_k \\ & = & (-1)^N (h_0 \lambda^n + h_1 \lambda^{n-1} + ... + h_n = (-1)^n h_\lambda, , \qquad \zeta_i = \beta_i - \lambda. \end{array} \tag{8}$$

This is the end of the citation with the only comment that the generating function could be obtained as irreducible representation of the symmetry group of the dressing chain equations (1) [11]. More general symmetry is studied in [9].

## 2 Explicit formulas for the solutions of the the chain equations $(N = 3)$

If the system (1) is interpreted as a result of factorization of the operator SL of the equation (2) and the relation (3) supplies the necessary condition for this. If, further the factorization is linked to the DT, then $\mu$ is the spectral parameter corresponding to the auxiliary spectral function that parameterized the transformation. The parameter $\alpha$ is zero in this case.

Let us consider equations

$$\begin{array}{rcl} \sigma_1' & = & \sigma_3^2 - \sigma_2^2 + \mu_3 - \mu_2, \\ \sigma_2' & = & \sigma_1^2 - \sigma_3^2 + \mu_1 - \mu_3, \\ \sigma_3' & = & \sigma_2^2 - \sigma_1^2 + \mu_2 - \mu_1, \end{array} \tag{9}$$

that is equivalent to the system (4) under the condition (3) with N=3. We use two integrals

$$\begin{array}{rcl} C & = & \sigma_1 + \sigma_2 + \sigma_3 \\ A & = & g_1 g_2 g_3 + \mu_2 g_3 + \mu_1 g_2 + \mu_3 g_1 = \\ & & (\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) + \\ & & \mu_2(\sigma_3 + \sigma_1) + \mu_1(\sigma_2 + \sigma_3) + \mu_3(\sigma_1 + \sigma_2) \end{array} \tag{10}$$

From (1) one obtains

$$\sigma_1'^2 = (\sigma_3^2 - \sigma_2^2 + \mu_3 - \mu_2)^2$$

Using the first equation (2) $\sigma_3 = C - \sigma_1 - \sigma_2$ we will obtain the equation containing $\sigma_1', \sigma_2$. As the next step we eliminate the remaining variable $\sigma_2$ with the help the second integral in (2) and the first also. Thus, we get the equation

$$\begin{aligned} \sigma_1'^2 &= \sigma_1^4 - 2(C^2 + \mu_3 + \mu_2 - 2\mu_1)\sigma_1^2 \\ &\quad -4(2\mu C - A)\sigma_1 + C^4 + 2(\mu_2 + \mu_3 + 2\mu_1)C^2 - 4AC + (\mu_3 - \mu_2)^2 \end{aligned} \tag{11}$$

The last equation has the form

$$\left(\frac{\sigma_1}{dx}\right)^2 = \sigma_1^4 - 6a\sigma_1^2 + 4b\sigma_1 + d, \tag{12}$$

where $a, b, d$ are constants defined by the previous expression (3). The extra multipliers 6,4 have been included for a convenience. The relation (12) is an elliptic curve in variables $(\sigma_1', \sigma_1)$ and therefore it is uniformised by elliptic functions. Let us built the invariants (capital letters are chosen that to distinguish them from variables $g_j$ of the chain)

$$G_2 = d + a^2 \qquad G_3 = a^3 - b^2 - ad$$

So, the pair $(b, a)$ is a point on a curve

$$b^2 = 4a^3 - G_2 a - G_3.$$

Therefore, there exists a parameter $\nu$ such that the following equations will be hold:

$$b = \wp'(2\nu), \qquad a = \wp(2\nu).$$

This means that we take three new parameters $G_2, G_3, \nu$ instead of old ones $a, b, d$ depended on five parameters of the chain: $(\mu_1, \mu_2, \mu_3, A, C)$. Now we may write

$$\left(\frac{d\sigma_1}{dx}\right)^2 = \sigma_1^4 - 6\wp(2\nu)\sigma_1^2 + 4\wp'(2\nu)\sigma_1 + (G_2 - 3\wp(2\nu)), \tag{13}$$

that yields

$$\sigma_1(x) = \zeta(x + \nu + x_0; G_2, G_3) - \zeta(x - \nu + x_0; G_2, G_3) - \zeta(2\nu; G_2, G_3).$$

Note, the $\sigma_1$ is not Weierstrass's $\sigma$-function in the theory of elliptic functions, but $\zeta$ is standard Weierstrass $\zeta$-function. The solution $\sigma_1(x)$ contains three arbitrary constants (according to the third order of equations (1)) $G_2, G_3, x_0$ which are, in its turn, defined by five ones $\mu_j, A, C$ in explicit but transcendental way. Parameter $\nu$ is not exceptional

$$\nu = \frac{1}{2}\wp^{-1}\left(\frac{\mu_3 + \mu_2 - 2\mu_1 + C^2}{3}\right),$$

where $\wp^{-1}$ denotes an elliptic integral of the first kind (inversion of elliptic function $\wp$).

**Remark 1:** Just obtained solution solution is exactly logarithmic derivative of the $\Psi$-function for the 1-gap Lame potential

$$\Psi'' - 2\wp(x)\Psi = \lambda\Psi \qquad \Psi = \frac{\sigma(\alpha - x)}{\sigma(\alpha)\sigma(x)} e^{\zeta(\alpha)x}, \qquad \lambda = \wp(\alpha).$$

and distinguished from the solution

$$\frac{\Psi'}{\Psi} = \zeta(\alpha - x) - \zeta(\alpha) - \zeta(x)$$

by the shift of the spectral parameter $\alpha$.

**Remark 2:** If one interests in $\sigma_i$ (12) in a connection with KdV equation theory, the dependence on time could be obtained using the t-chains [11], obtained by means of MKdV equation for $\sigma$ and conservation laws [10].

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